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consideration: Historical Overview of Common Goniometrical Scales
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Goniometric Scales on Generic Slide Rules

Introduction

In the history of the slide rule we can distinguish between generic and specialist slide rules.
Goniometric scales occur in many forms on specialist slide rules, like those for navigation or geodesy.
This article however will focus on the design of goniometric scales for generic slide rules, which were intended for general use, in any branch or craft.
Sines and tangents were the foremost used goniometrical proportions at the dawn of the slide rule: the older “Chord” function was only used in construction and drawing, while derived functions like cosine, secans etc. would gain acceptance only later. The units in which angles can be expressed, are some of the least standardised in the world.
The 360 degrees for a “full circle” angle is the oldest and most used. But the sexagesimal subdivision in minutes and seconds, as used by navigators and astronomers, was not very handy in calculations.
Therefore we see two systems in generic slide rules. The first one uses the subdivision of one degree into 60 minutes (but replaces seconds by decimal fractions of the minute). The second -and more modern- approach is the use of decimal degrees, where minutes and seconds are replaced by decimal fractions of the degree: this last system had the self-explanatory name “DeciTrig” in slide rules of Keuffel & Esser. Users had to be very aware of this system when reading scales. Here we’ll use decimal degrees.
Many other units exist for angles, like the 400^g “grad” system in geodesy, the 32 “Rhumbs” or compass points of navigators, the Mil’s of the military, the radians of mathematicians, and much more. All these can be found on specialist slide rules.

Gunter’s Lines

The Gunter scales were published by Edmund Gunter in 1624, and can be considered as the “Mother of All Slide Rules”, although the “slide” function had to be applied by moving distances with a pair of dividers.
With the three basic scales N (the logarithmical Line of Numbers), S (the logarithmical Line of Sines), and T (the logarithmical Line of Tangents) any multiplication, division, goniometrical and/or trigonometrical calculation could be executed, also in combinations.
Figure 1 depicts Gunter’s 2-cycle scales, although the left part of the first cycle is cut off. The beauty lies in the fact that number, sine and tangent scales are smoothly interworking: the only flaw is that the Num line might have had readings from 0.01 thru 0.1 to 1.

The power of Gunter’s design is that he proposed already a two-cycle logarithmic scale N, so that his Sine scale ranged from 0.573º to 90º, because sin (0.573º) = 0.01 and sin (90º) = 1.
For the Tangent scale his range was from 0.573º to 45º.
In principle these ranges could be extended to smaller values by adding cycles to the left of the N-scale, or to higher values -in case of tangents- by adding cycles to the right.
Figure 2 indicates how the ranges of the sine and the tangent scales can be extended by providing more cycles to the Line of Number.

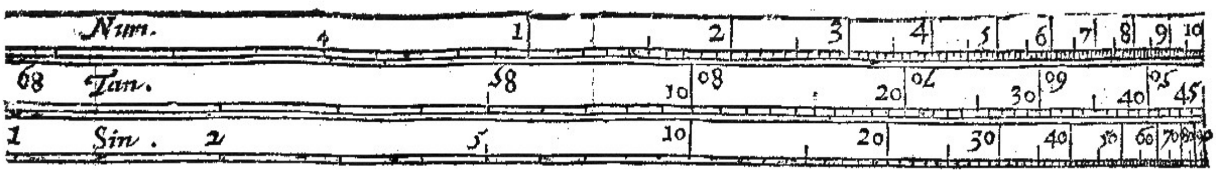


Figure 1: Original drawing of Gunter's Scales

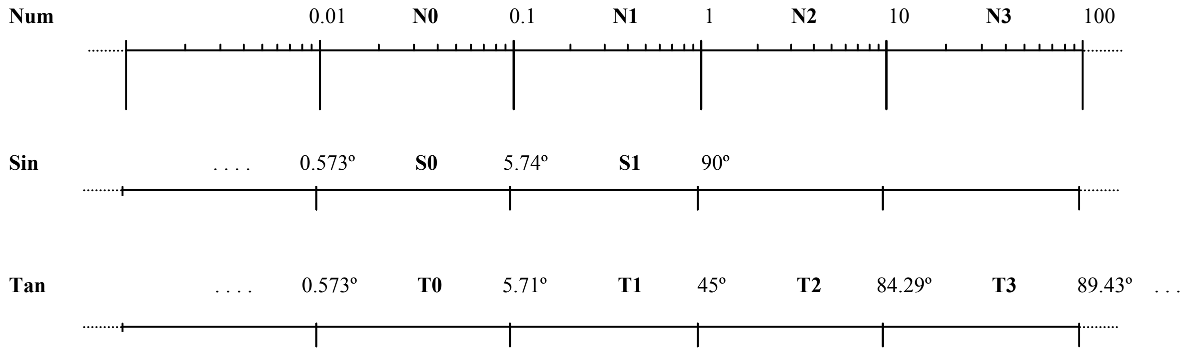


Figure 2: Goniometric ranges and number of cycles

Oughtred's Circles

Oughtred's circular calculator, which he called "The Circles of Proportion", was the first generic slide rule constructed as a disc with two index pointers moving over the circular scales either independently or together.

Although Oughtred's circular instrument has received much attention as the first slide rule ever, the scales on his disc are not widely known. Oughtred acknowledges Gunter's scales when he describes that his instrument "*only bows and inflects Gunter's lines*". But his design was more extensive.

His scales (which are to be read anti-clockwise) can be identified as follows, working from outside to inside, see fig. 3:

1. S (Sines from 5.74º to 90º)
2. T (Tangents from 5.71º to 45º)
3. T (Tangents from 45º to 84.29º)
4. N (Line of Numbers from 1 to 10)
5. E (Logarithms base 10 from 0 to 10)
6. T (Tangents from 84.29º to 89.43º)
7. T (Tangents from 0.573º to 5.71º)
8. S (Sines from 0.573º to5.74º)

There are some more inner circles, which are related to astronomical applications. Considering these scales, we can conclude that Oughtred's instrument was in the first place a goniometrical calculator, consisting mainly of sine and tangent scales ranging from a half to some 90 degrees. This takes 4 cycles, but thanks to the endless circle only one cycle is needed in the 4th scale, the Line of Numbers.

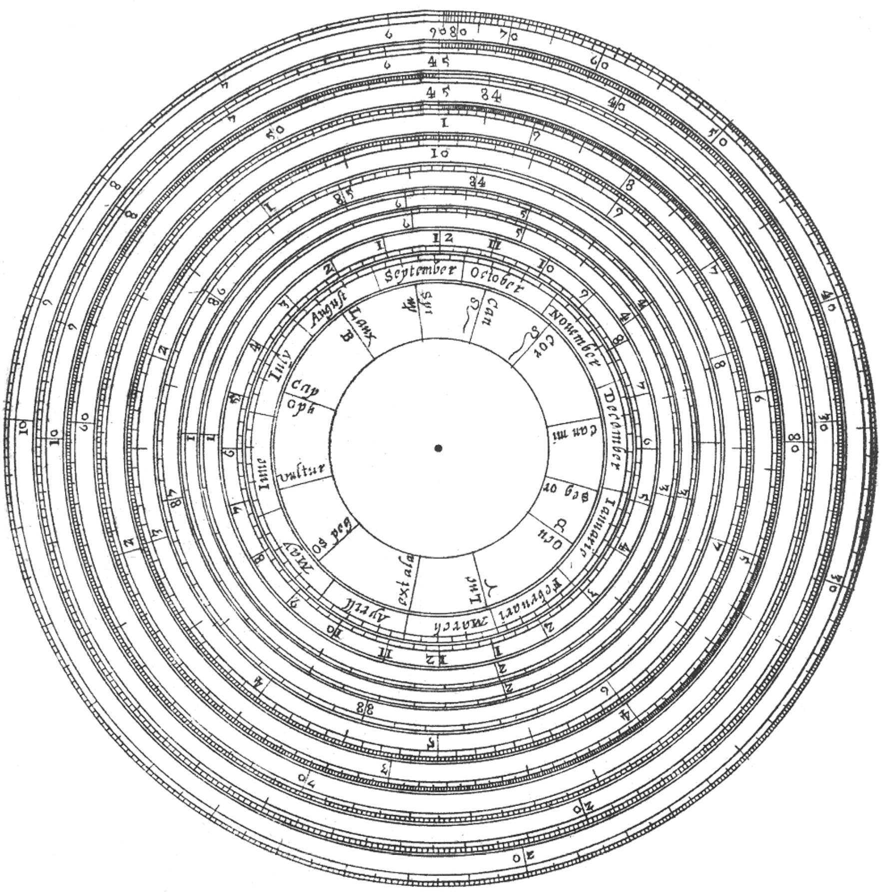


Figure 3: Oughtred's Scales (from Journal of the Oughtred Society, Vol. 2, No. 2, Fall 2001)

After Oughtred

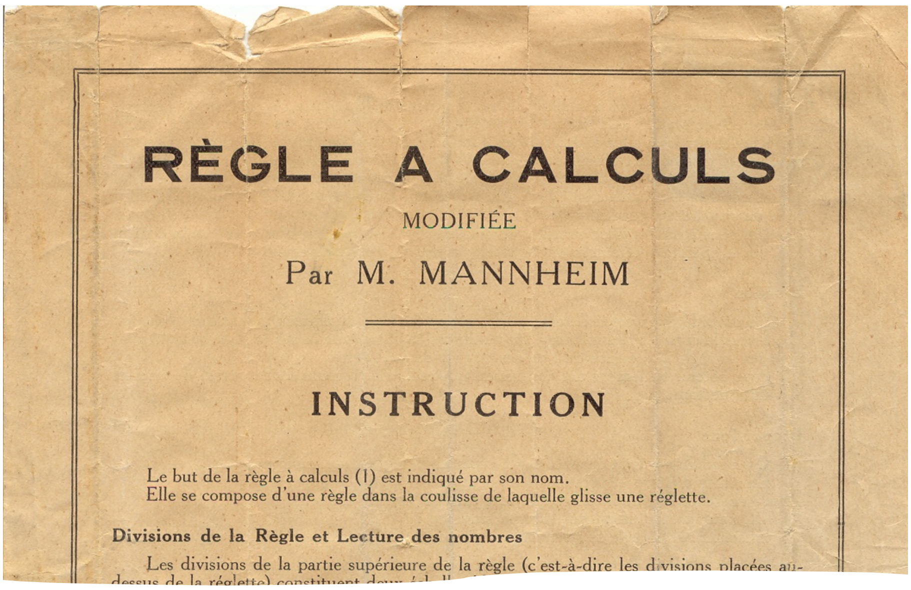
After Oughtred we skip a large number of slide rule designs, mostly for special trades like gauging or carpentry, until we see in the late 18th century the generic **SOHO**-rule designed by James Watt.
Famous though it was, with its cursorless "**A = B B = D**" scale design, we dismiss it for our purposes since it had no goniometric scales at all.
Many more slide rules have been missing goniometric scales altogether, for example most cylindrical types, and also the very simple school models or pocket-sized ones.
Then there was a class of "fake" goniometrical scales, meaning that scales for sines and tangents did occur, but could not be connected with the number scales A/B or C/D.
For example, many "Binary" circular slide rules did have such unconnected sine and tangent tables on one side, only fit for table look-up.

Mannheim

In the mid 19th century the Mannheim design originated in France, with the cursor re-introduced. Mannheim introduced the **A = B C = D** scale design which would last for the rest of the slide rule era.
Looking for goniometric scales, we are disappointed to see them banned to the back of the slide; for some reason Mr. Mannheim failed to realize the full benefits of a vertical hairline, when he did not add sine and tangent scales to the front of the slide rule, but to the back of the slide.

Both the sine and tangent scale were relating to the two-cycle A/B scale, just like the original Gunter scales, and therefore had the S0-S1 and the T0-T1 range according to figure 2. In a 4-page "*Instruction, Règle A Calculs, modifié par M.Mannheim*", see fig. 4, we read that the procedure required one to take out the slide and re-insert with the goniometric scales in front: then the sine scale was adjacent to A.
If a calculation with tangents came up, a new reinsertion was needed, but now upside-down to get the tangent scale adjacent to A. That is the reason for the upside-down T-scale, see figure 5.

This is called "method I" in figure 6, which summarizes the successive approaches of the goniometric scales.


RÈGLE A CALCULS

MODIFIÉE

Par M. MANNHEIM

INSTRUCTION

Le but de la règle à calculs (1) est indiqué par son nom.
Elle se compose d'une règle dans la coulisse de laquelle glisse une réglette.

Divisions de la Règle et Lecture des nombres

Les divisions de la partie supérieure de la règle (c'est-à-dire les divisions placées au-

Figure 4: Mannheim's 4-page Instruction leaflet

Amédée Mannheim wrote in his instructions on the goniometrical scales:

Sinus, Tangentes
L'échelle S du revers de la réglette est l'échelle des sinus. Les longueurs, comptées à partir de l'extrémité gauche de cette échelle jusqu'à 1, 2, 3, etc., représentent les logarithmes des sinus naturels des angles de 1º, 2º, 4º, etc., mesurés dans une circonférence de rayon 100.
Le dernier trait à droite correspond à sin. 90º.
Le premier trait à droite, à l'extrémité gauche de l'échelle, correspond à sin. 35´.
L'échelle T, l'échelle des tangentes, est construite de la même manière.
On met l'échelle dont on veut se servir en contact avec l'échelle supérieure de la règle.
Si l'on fait coïncider les extrémités des échelles S ou T avec les extrémités de l'échelle supérieure, on lira en face des traits 1, 2, 3, etc., les sinus ou les tangentes de ces angles.
En attribuant au trait 1 gauche de l'échelle supérieure de la règle la valeur 0,01 et par suite, au trait 1 droite la valeur 1, on aura la valeur de ces lignes trigonométriques dans une circonférence de rayon 1.
On obtiendra la valeur des tangentes des angles plus grands que 45º en divisant 1 par la tangente de l'angle complémentaire.
Ces échelles sont employées dans les calculs où il entre des lignes trigonométriques de la même manière que les échelles ordinaires des nombres : Soit 38 x sin. 15º ; amener l'une des extrémités de l'échelle des sinus sous 38, lire le produit sur l'échelle supérieure de la règle, en regard du trait correspondant à sin. 15º.
Sur la première échelle de la réglette on a placé deux traits indiquant les nombres 343 et 206205. Le premier, accompagné de ´ , correspond au logarithme de 1 / (sin. 1´) ; le second, accompagné de ", correspond au logarithme de 1 / (sin. 5´). En admettant que les sinus des angles très petits soient proportionnels au angles, on obtiendra facilement les sinus des angles très petits au moyen de ces traits.
Soit à chercher sin. 19´ : on placera en regard du nombre 19 lu sur l'échelle supérieure de la règle le trait accompagné de ´. On lira au-dessus du 1, milieu de l'échelle de la réglette, la valeur de sin. 19´. Si l'on avait à effectuer le produit de sin. 19´ par 4, on lirait immédiatement ce produit au-dessus du 4 sur l'échelle supérieure de la réglette.

For practical usage, this re-insertion scheme was definitely less user-friendly than the ancient designs of Gunter and Oughtred.
It is fair to mention that one could also calculate without reinserting the slide, provided the back of the body had left and right windows with index markers: then an S- or T-value could be set at value 1 of the A or D scale. However this "invisible" use of sine and tangent scales was more cumbersome, and the precision of the backside index markers was less than a sine and tangent scale on the front would have offered.

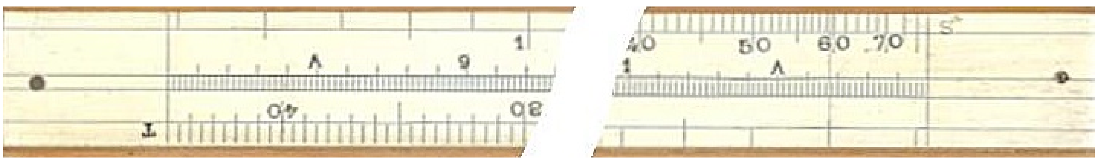


Figure 5: Back of Mannheim slide, with opposing S and T scale

Further Developments

A new development took place in the early 20th century by German slide rule producers. It was felt that goniometric scales would increase in precision if they were related to the one-cycle C/D scale, in stead of the two-cycle A/B scale. So the sine scale remained bound to A/B, but the tangent scale related to the more precise C/D scale. The result of course was that the tangent range was narrowed to T1 only. But, as it happened, the lower sine range (S0) could also be used as an approximation for lower tangents although the difference can be noticed. This is called "method II" in figure 6.
For actual use this was a small improvement because after re-inserting the reversed slide, the sine was adjacent to A and the tangent scale to D. Thus both functions could occur in calculations (although not on the same scale level). Still the slide reversal had to be undone every time a common A-B or C-D operation was needed: chain calculations with goniometric functions and numbers were practically impossible.

With the German **Rietz** design, the trend was drawn to its conclusion around 1905: both sine and tangent scale now related to the C scale, thereby losing the low value range S0 and T0. To allow for lower range angles, a new ST scale was soon introduced which related an "approximate" sine and tangent average to the "precise" C scale. This is called "method III" in figure 6. The ST scale had different implementations: some were an exact sine scale, others really used an average between sine and tangent values. But I have also found a slide rule (Duval) with tangent values on the ST scale.

In the area of 5º the ST error could be over 0.3 % which is worse than the general slide rule precision. For low angle values there is actually no need for a separate sine or tangent scale: in that range the sine (and tangent) function can be approximated by sin (r) = r, where r is expressed in radians. Already the Mannheim "Instruction" described a ´ gauge mark, later called ϱ´ (ϱ´ = 360 x 60 / 2π = 3.438), which allowed calculating sine or tangent by dividing the angle (expressed in minutes) by ϱ´. The error produced by this method in the ST range of fig. 2 is for the sine function lower than by using the averaged ST scale, and for the tangent it is comparable.
There were equivalents of the ϱ´ system, to allow use of seconds (ϱ´´), or decimal degrees (ϱ).

A next system, **Darmstadt**, also had consequences for goniometrical calculations: because in most versions the back of the slide was used for the new log-log scales, the goniometric scales moved to the body (stator), in many cases on the side surface. This is called "method IV" in figure 6.

It was better than the "backward" Mannheim, but in the process the ST scale was left out due to lack of space: so now both sines and tangents missed the lower values that ST had provided in a way, but this was replaced on most Darmstadt models by the system of ϱ marks.

S and T Ranges versus Scales	< 0.573º	S0 (0.573º - 5.74º)	S1 (5,74º-90º)	T0 (0.573º - 5.71º)	T1 (5.71º-45º)	T2 (45º-84.29º)	T3 (84.29º - 89.43º)
Gunter (1624)	-	**SIN to NUM**	**SIN to NUM**	**TAN to NUM**	**TAN to NUM**	use 1/cotan	use 1/cotan
Oughtred (about 1630)	-	**Scale 8 to N**	**Scale 1 to N**	**Scale 7 to N**	**Scale 2 to N**	**Scale 3 to N**	**Scale 6 to N**
Method I (Mannheim, 1851)	use ρ marks (if present)	**S to A**	**S to A**	**T to A**	**T to A**	use 1/cotan	use 1/cotan
Method II	use ρ marks (if present)	**S to A**	**S to A**	use ρ marks (if present)	**T to C**	use 1/cotan	use 1/cotan
Method III (Rietz, 1905)	use ρ marks (if present)	**ST to C**	**S to C**	**ST to C**	**T to C**	use 1/cotan	use 1/cotan
Method IV (Darmstadt, 1935)	use ρ marks (if present)	use ρ marks (if present)	**S to D**	use ρ marks (if present)	**T to D**	use 1/cotan	use 1/cotan
Method V (Duplex, postwar)	use ρ marks (if present)	**ST to C/D**	**S to C/D**	**ST to C/D**	**T1 to C/D**	**T2 to C/D**	use 1/cotan

Figure 6: Between which scales (shaded cells) the various sine and tangent ranges are accommodated

Duplex Slide Rules

The Darmstadt had grown so many scales that further extensions really needed the duplex design (a two-sided slide rule) which almost doubled the space available for scales. The additional space on a Duplex was used for increased flexibility (folded scales CF/DF), or for increased precision (split C/D scales or additional square roots W or R), or for extended ranges of the log-log scales, or for easier use in chain calculations (many kinds of inverted I-scales on slide and body).

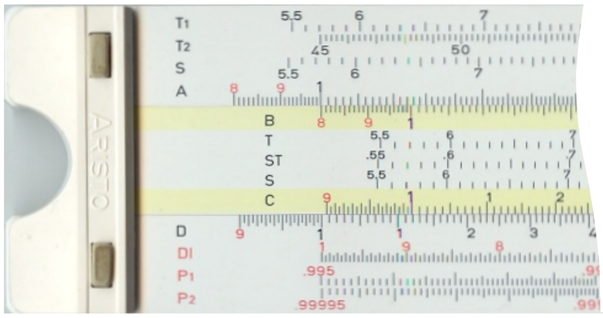
When all these improvements were added together, a huge slide rule resulted, which most manufacturers could provide during the 1960's and 1970's (for example Faber's Novo-Duplex 2/83N, Aristo's StudioLog 0969, K&E's Decilon 681100 or Dietzgen's Versalog 1460).
One of the possible extensions was a second tangent scale for the range T2, and the return of the ST scale. This is called "method V" in figure 6.
Before T2 appeared on the scene, all tangents of values x between 45º and 90º had to be calculated on T1 via the clumsy workaround 1 / cotan (x) .

Chain Calculations

A more complex formula is calculated by a chain of slide rule operations, where the challenge is to use a minimum of slide movements. Some users think first and do some rearranging of operations before starting calculations. Others start unthinking at the first operation and wish to slide thru to the end, in an unprepared sequence. But sometimes an intermediate result cannot be used as input for the next operation. For example, the intermediate result happens to show on slide scale C, at D = 1. Then the slide can not be moved for a next operation, which is just needed when next a multiplication with, say, sin(37º) is needed, assuming the S scale is on the slide too. That is the reason why the duplex rules also included many inverted scales to allow multiplication via inverse division. But not for goniometric scales: on most of my duplexes, S, ST, T1, T2 are either all on the slide (the majority), or all on the body. Only a few had goniometric scales on both slide and body, for example Aristo's "MultiTrig" 0929, see fig. 7.

The advanced new scales on duplexes did not help very much for the goniometric scales because many of them (like the folded and the split scales) could not interwork with the sine and tangent scales.

Figure 7: Multiple S and T scales on Aristo MultiTrig



Alternative Approach

One manufacturer should be mentioned for a new idea on goniometric scales. PIC in England has designed the "Differential Scales" for sines and tangents, for example on the PIC No. 121 /PC18.

For the sine function a short scale (called sine differential: SD) is added to the slide representing the value "x / sin (x)" on the C scale. When "x" on the D scale is divided by "x / sin (x)" on the C scale, the result of the division "sin (x)" can be read on the D scale under C=1 or C=10, see fig. 8.

The claimed advantage is smaller scale length for a larger range in degrees, but this goes at the cost of decreased precision. It is true that close to 90º the resolution on the SD scale is better than with a classical S scale, but that goes at the cost of lost precision below 20º.
PIC's Differential Scales found no followers.

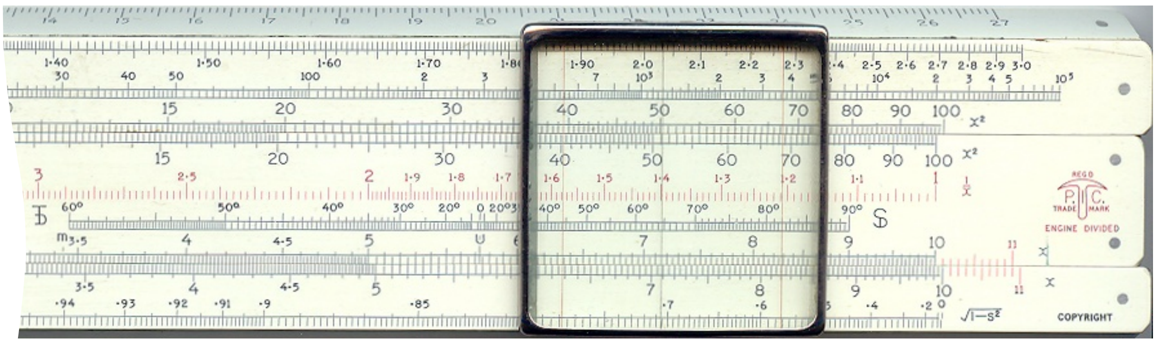


Figure 8: PIC's Differential S and T Scales, showing by the rightmost hairline: sin (82º) = 0.990

Conclusion

Gunter's scales and Oughtred's Circles were innovative goniometrical machines in the early 1600's.
Oughtred's disc had an impressive range and precision, even compared to modern slide rules.

The Mannheim rule re-introduced Gunter's sine and tangent scales (alas on the back of the slide), but also introduced the powerful ϱ markings for small angles. Since then no real progress has been made in goniometric scales on generic slide rules.
For more elaborate goniometric calculations, specialist slide rules were needed like those designed for navigation or geodesy.